the backward cascade process. This approach can therefore be classed in the category of statistical deterministic backward cascade models.

Representing the backward cascade by way of a negative viscosity is controversial because the theoretical analyses, such as by the EDQNM model, distinguish very clearly between the cascade and backward cascade terms, both in their intensity and in their mathematical form [192, 193]. This representation is therefore to be linked to other statistical deterministic descriptions of the backward cascade, which take into account only an average reduction of the effective viscosity, such as the Chollet–Lesieur effective viscosity spectral model.

The main backward cascade models belonging to these two categories are described in the following.

4.4.2 Deterministic Statistical Models

This section describes the deterministic models for the backward cascade. These models, which are based on a modification of the subgrid viscosity associated with the forward cascade process, are:

1. The spectral model based on the theories of turbulence proposed by Chasnov (p.126). A negative subgrid viscosity is computed directly from the EDQNM theory. No hypothesis is adopted concerning the spectrum shape of the resolved scales, so that the spectral disequilibrium mechanisms can be taken into account at the level of these scales, but the spectrum shape of the subgrid scales is set arbitrarily. Also, the filter is assumed to be of the sharp cutoff type.
2. The dynamic model with an equation for the subgrid kinetic energy (p.127), to make sure this energy remains positive. This ensures that the backward cascade process is represented physically, in the sense that a limited quantity of energy can be restored to the resolved scales by the subgrid modes. However, this approach does not allow a correct representation of the spectral distribution of the backward cascade. Only the quantity of restored energy is controlled.

Chasnov's Spectral Model. Chasnov [54] adds a model for the backward cascade, also based on an EDQNM analysis, to the cascade model already described (see Sect. 4.3.1). The backward cascade process is represented deterministically by a negative effective viscosity term $\nu_e^-(k|k_c)$, which is of the form:

$$\nu_e^-(k|k_c,t) = -\frac{F^-(k|k_c,t)}{2k^2E(k,t)} \quad . \tag{4.233}$$

The stochastic forcing term is computed as:

$$F^-(k|k_c,t) = \int_{k_c}^{\infty} dp \int_{p-k}^{p} dq \Theta_{kpq} \frac{k^3}{pq}(1-2x^2z^2-xyz)E(q,t)E(p,t), \tag{4.234}$$

in which x, y, and z are geometric factors associated with the triad $(\mathbf{k},\mathbf{p},\mathbf{q})$, and Θ_{kpq} is a relaxation time described in Appendix B. As is done when computing the draining term (see Chasnov's effective viscosity model in Sect. 4.3.1), we assume that the spectrum takes the Kolmogorov form beyond the cutoff k_c. To simplify the computations, formula (4.234) is not used for wave numbers $k_c \leq p \leq 3k_c$. For the other wave numbers, we use the asymptotic form

$$F^-(k|k_c,t) = \frac{14}{15}k^4 \int_{k_c}^{\infty} dp \Theta_{kpp}(t)\frac{E^2(p,t)}{p^2} \quad . \tag{4.235}$$

This expression complete Chasnov's spectral subgrid model which, though quite close to the Kraichnan type effective viscosity models, makes it possible to take into account the backward cascade effects that are dominant for very small wave numbers.

Localized Dynamic Model with Energy Equation. The Germano–Lilly dynamic procedure and the localized dynamic procedure lead to the definition of subgrid models that raise numerical stability problems because the model constant can take negative values over long time intervals, leading to exponential growth of the disturbances.

This excessive duration of the dynamic constant in the negative state corresponds to too large a return of kinetic energy toward the large scales [47]. This phenomenon can be interpreted as a violation of the spectrum realizability constraint: when the backward cascade is over-estimated, a negative kinetic energy is implicitly defined in the subgrid scales. A simple idea for limiting the backward cascade consists in guaranteeing spectrum realizability[24]. The subgrid scales cannot then restore more energy than they contain. To verify this constraint, local information is needed on the subgrid kinetic energy, which naturally means defining this as an additional variable in the simulation.

A localized dynamic model including an energy equation is proposed by Ghosal *et al.* [122]. Similar models have been proposed independently by Ronchi *et al.* [233, 281] and Wong [348]. The subgrid model used is based on the kinetic energy of the subgrid modes. Using the same notation as in Sect. (4.3.3), we get:

$$\alpha_{ij} = -2\widetilde{\overline{\Delta}}\sqrt{Q_{\mathrm{sgs}}^2}\widetilde{\overline{S}}_{ij} \quad , \tag{4.236}$$

$$\beta_{ij} = -2\overline{\Delta}\sqrt{q_{\mathrm{sgs}}^2}\overline{S}_{ij} \quad , \tag{4.237}$$

in which the energies Q_{sgs}^2 and q_{sgs}^2 are defined as:

[24] The spectrum $E(k)$ [illegible]id to be realizable if $E(k) \geq 0$, $\forall k$.

$$Q^2_{\rm sgs} = \frac{1}{2}\left(\widetilde{\overline{u_i u_i}} - \tilde{\overline{u}}_i\tilde{\overline{u}}_i\right) = \frac{1}{2}T_{ii} \quad , \tag{4.238}$$

$$q^2_{\rm sgs} = \frac{1}{2}\left(\overline{u_i u_i} - \overline{u}_i\overline{u}_i\right) = \frac{1}{2}\tau_{ii} \quad . \tag{4.239}$$

Germano's identity (4.126) is written:

$$Q^2_{\rm sgs} = \widetilde{q^2_{\rm sgs}} + \frac{1}{2}L_{ii} \quad . \tag{4.240}$$

The model is completed by calculating $q^2_{\rm sgs}$ by means of an additional evolution equation. We use the equation already used by Schumann, Horiuti, and Yoshizawa, among others (see Sect. 4.3.2):

$$\frac{\partial q^2_{\rm sgs}}{\partial t} + \frac{\partial \overline{u}_j q^2_{\rm sgs}}{\partial x_j} = -\tau_{ij}\overline{S}_{ij} - C_1\frac{(q^2_{\rm sgs})^{3/2}}{\overline{\Delta}}$$
$$+C_2\frac{\partial}{\partial x_j}\left(\overline{\Delta}\sqrt{q^2_{\rm sgs}}\frac{\partial q^2_{\rm sgs}}{\partial x_j}\right) + \nu\frac{\partial^2 q^2_{\rm sgs}}{\partial x_j \partial x_j}, \tag{4.241}$$

in which the constants C_1 and C_2 are computed by a constrained localized dynamic procedure described above. The dynamic constant $C_{\rm d}$ is computed by a localized dynamic procedure.

This model ensures that the kinetic energy $q^2_{\rm sgs}$ will remain positive, *i.e.* that the subgrid scale spectrum will be realizable. This property ensures that the dynamic constant cannot remain negative too long and thereby destabilize the simulation. However, finer analysis shows that the realizability conditions concerning the subgrid tensor τ (see Sect. 3.3.5) are verified only on the condition:

$$-\frac{\sqrt{q^2_{\rm sgs}}}{3\overline{\Delta}|s_\gamma|} \le C_{\rm d} \le \frac{\sqrt{q^2_{\rm sgs}}}{3\overline{\Delta}s_\alpha} \quad , \tag{4.242}$$

where s_α and s_γ are, respectively, the largest and smallest eigenvalues of the strain rate tensor $\overline{S}$. The model proposed therefore does not ensure the realizability of the subgrid tensor.

The two constants C_1 and C_2 are computed using an extension of the constrained localized dynamic procedure. To do this, we express the kinetic energy $Q^2_{\rm sgs}$ evolution equation as:

$$\frac{\partial Q^2_{\rm sgs}}{\partial t} + \frac{\partial \tilde{\overline{u}}_j Q^2_{\rm sgs}}{\partial x_j} = -T_{ij}\tilde{\overline{S}}_{ij} - C_1\frac{(Q^2_{\rm sgs})^{3/2}}{\tilde{\overline{\Delta}}}$$
$$+ C_2\frac{\partial}{\partial x_j}\left(\sqrt{Q^2_{\rm sgs}}\frac{\partial Q^2_{\rm sgs}}{\partial x_j}\right) + \nu\frac{\partial^2 Q^2_{\rm sgs}}{\partial x_j \partial x_j} . \tag{4.243}$$

One variant of the Germano's relation relates the subgrid kinetic energy flux f_j to its analog at the level of the test filter F_j:

$$F_j - \widetilde{f_j} = Z_j \equiv \tilde{\overline{u}}_j(\overline{p} + \widetilde{q^2_{\rm sgs} + \overline{u}_i\overline{u}_i}/2) - \widetilde{\overline{u}_j(\overline{p} + q^2_{\rm sgs}} + \overline{u}_i\overline{u}_i/2) \quad , \tag{4.244}$$

in which $\overline{p}$ is the resolved pressure.

To determine the constant C_2, we substitute in this relation the modeled fluxes:

$$f_j = C_2\overline{\Delta}\sqrt{q^2_{\rm sgs}}\frac{\partial q^2_{\rm sgs}}{\partial x_j} \quad , \tag{4.245}$$

$$F_j = C_2\tilde{\overline{\Delta}}\sqrt{Q^2_{\rm sgs}}\frac{\partial Q^2_{\rm sgs}}{\partial x_j} \quad , \tag{4.246}$$

which leads to:

$$Z_j = X_j C_2 - \widetilde{Y_j C_2} \quad , \tag{4.247}$$

in which

$$X_j = \tilde{\overline{\Delta}}\sqrt{Q^2_{\rm sgs}}\frac{\partial Q^2_{\rm sgs}}{\partial x_j} \quad , \tag{4.248}$$

$$Y_j = \overline{\Delta}\sqrt{q^2_{\rm sgs}}\frac{\partial q^2_{\rm sgs}}{\partial x_j} \quad . \tag{4.249}$$

Using the same method as was explained for the localized dynamic procedure, the constant C_2 is evaluated by minimizing the quantity:

$$\int\left(Z_j - X_j C_2 + \widetilde{Y_j C_2}\right)\left(Z_j - X_j C_2 + \widetilde{Y_j C_2}\right) \quad . \tag{4.250}$$

By analogy with the preceding developments, the solution is obtained in the form:

$$C_2(\mathbf{x}) = \left[f_{C_2}(\mathbf{x}) + \int \mathcal{K}_{C_2}(\mathbf{x},\mathbf{y})C_2(\mathbf{y})d^3\mathbf{y}\right]_+ \quad , \tag{4.251}$$

in which:

$$f_{C_2}(\mathbf{x}) = \frac{1}{X_j(\mathbf{x})X_j(\mathbf{x})}\left(X_j(\mathbf{x})Z_j(\mathbf{x}) - Y_j(\mathbf{x})\int Z_j(\mathbf{y})G(\mathbf{x}-\mathbf{y})d^3\mathbf{y}\right) \quad , \tag{4.252}$$

$$\mathcal{K}_{C_2}(\mathbf{x},\mathbf{y}) = \frac{\mathcal{K}^{C_2}_{\mathcal{A}}(\mathbf{x},\mathbf{y}) + \mathcal{K}^{C_2}_{\mathcal{A}}(\mathbf{y},\mathbf{x}) - \mathcal{K}^{C_2}_{\mathcal{S}}(\mathbf{x},\mathbf{y})}{X_j(\mathbf{x})X_j(\mathbf{x})} \quad , \tag{4.253}$$

in which

$$\mathcal{K}_{\mathcal{A}}^{C_2}(\mathbf{x},\mathbf{y}) = X_j(\mathbf{x})Y_j(\mathbf{y})G(\mathbf{x}-\mathbf{y}) \quad , \tag{4.254}$$

$$\mathcal{K}_{S}^{C_2}(\mathbf{x},\mathbf{y}) = Y_j(\mathbf{x})Y_j(\mathbf{y})\int G(\mathbf{z}-\mathbf{x})G(\mathbf{z}-\mathbf{y})d^3\mathbf{z} \quad . \tag{4.255}$$

This completes the computation of constant C_2. To determine the constant C_1, we substitute (4.240) in (4.243) and get:

$$\frac{\partial \widetilde{q^2_{\mathrm{sgs}}}}{\partial t} + \frac{\partial \widetilde{\overline{u}}_j \widetilde{q^2_{\mathrm{sgs}}}}{\partial x_j} = -E\frac{\partial F_j}{\partial x_j} + \nu \frac{\partial^2 \widetilde{q^2_{\mathrm{sgs}}}}{\partial x_j\partial x_j} \quad , \tag{4.256}$$

in which E is defined as:

$$E = T_{ij}\widetilde{\overline{S}}_{ij} + \frac{C_1 (Q^2_{\mathrm{sgs}})^{3/2}}{\widetilde{\overline{\Delta}}} - \nu\frac{1}{2}\frac{\partial^2 L_{ii}}{\partial x_j \partial x_j} + \frac{1}{2}\left(\frac{\partial L_{ii}}{\partial t} + \frac{\partial \widetilde{\overline{u}}_j L_{ii}}{\partial x_j}\right) \quad . \tag{4.257}$$

Applying the test filter to relation (4.241), we get:

$$\frac{\partial \widetilde{q^2_{\mathrm{sgs}}}}{\partial t} + \frac{\partial \widetilde{\overline{u}}_j\widetilde{q^2_{\mathrm{sgs}}}}{\partial x_j} = -\widetilde{\tau_{ij}\overline{S}_{ij}} - \left(C_1\frac{\widetilde{(q^2_{\mathrm{sgs}})^{3/2}}}{\overline{\Delta}}\right) + \frac{\partial \widetilde{f}_j}{\partial x_j} + \nu\frac{\partial^2 \widetilde{q^2_{\mathrm{sgs}}}}{\partial x_j\partial x_j} \quad . \tag{4.258}$$

By eliminating the term $\partial\widetilde{q^2_{\mathrm{sgs}}}/\partial t$ between relations (4.256) and (4.258), then replacing the quantity $F_j - \widetilde{f}_j$ by its expression (4.244) and the quantity T_{ij} by its value as provided by the Germano identity, we get:

$$\chi = \phi C_1 - \widetilde{\psi C_1} \quad , \tag{4.259}$$

in which

$$\chi = \widetilde{\tau_{ij}\overline{S}_{ij}} - \widetilde{\tau}_{ij}\widetilde{\overline{S}}_{ij} - L_{ij}\widetilde{\overline{S}}_{ij} + \frac{\partial \rho_j}{\partial x_j} - \frac{1}{2}D_t L_{ii} + \frac{1}{2}\nu\frac{\partial^2 L_{ii}}{\partial x_j\partial x_j} \quad , \tag{4.260}$$

$$\phi = (Q^2_{\mathrm{sgs}})^{3/2}/\widetilde{\overline{\Delta}} \quad , \tag{4.261}$$

$$\psi = (q^2_{\mathrm{sgs}})^{3/2}/\overline{\Delta} \quad , \tag{4.262}$$

and

$$\rho_j = \widetilde{\overline{u}}_j(\widetilde{\overline{p} + \overline{u}_i\overline{u}_i/2}) - \overline{u}_j(\widetilde{\overline{p} + \overline{u}_i\overline{u}_i/2}) \quad . \tag{4.263}$$

The symbol D_t designates the material derivative $\partial/\partial t + \widetilde{\overline{u}}_j\partial/\partial x_j$. The constant C_1 is computed by minimizing the quantity

$$\int\left(\chi - \phi C_1 + \widetilde{\psi C_1}\right)\left(\chi - \phi C_1 + \widetilde{\psi C_1}\right) \quad , \tag{4.264}$$

by a constrained localized dynamic procedure, which is written:

$$C_1(\mathbf{x}) = \left[f_{C_1}(\mathbf{x}) + \int \mathcal{K}_{C_1}(\mathbf{x},\mathbf{y})C_1(\mathbf{y})d^3\mathbf{y}\right]_+ \quad , \tag{4.265}$$

in which

$$f_{C_1}(\mathbf{x}) = \frac{1}{\phi(\mathbf{x})\phi(\mathbf{x})}\left(\phi(\mathbf{x})\chi(\mathbf{x}) - \psi(\mathbf{x})\int\chi(\mathbf{y})G(\mathbf{x}-\mathbf{y})d^3\mathbf{y}\right) \quad , \tag{4.266}$$

$$\mathcal{K}_{C_1}(\mathbf{x},\mathbf{y}) = \frac{\mathcal{K}_{\mathcal{A}}^{C_1}(\mathbf{x},\mathbf{y}) + \mathcal{K}_{\mathcal{A}}^{C_1}(\mathbf{y},\mathbf{x}) - \mathcal{K}_{S}^{C_1}(\mathbf{x},\mathbf{y})}{\phi(\mathbf{x})\phi(\mathbf{x})} \quad , \tag{4.267}$$

in which

$$\mathcal{K}_{\mathcal{A}}^{C_1}(\mathbf{x},\mathbf{y}) = \phi(\mathbf{x})\psi(\mathbf{y})G(\mathbf{x}-\mathbf{y}) \quad , \tag{4.268}$$

$$\mathcal{K}_{S}^{C_1}(\mathbf{x},\mathbf{y}) = \psi(\mathbf{x})\psi(\mathbf{y})\int G(\mathbf{z}-\mathbf{x})G(\mathbf{z}-\mathbf{y})d^3\mathbf{z} \quad , \tag{4.269}$$

which completes the computation of the constant C_1.

4.4.3 Stochastic Models

Models belonging to this category are based on introducing a random forcing term into the momentum equations. It should be noted that this random character does not reflect the space-time correlation scales of the subgrid fluctuations, which limits the physical validity of this approach and can raise numerical stability problems. It does, however, obtain forcing term formulations at low algorithmic cost. The models described here are:

1. Bertoglio's model in the spectral space (p.132). The forcing term is constructued using a stochastic process, which is designed in order to induce the desired backward energy flux and to possess a finite correlation time scale. This is the only random model for the backward cascade deerived in the spectral space.
2. Leith's model (p.133). The forcing term is represented by an acceleration vector deriving from a vector potential, whose amplitude is evaluated by simple dimensional arguments. The backward cascade is completely decoupled from forward cascade here: there is no control on the realizability of the subgrid scales.